NONHOLONOMIC STRATIFIED MOTION PLANNING
ALONG DECOMPOSED REFERENCE TRAJECTORIES

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Abstract
This work studies a new stratified motion planning algorithm whose mathematical details is established in [1]. The proposed technique is able to lead a stratified system along a restricted class of reference trajectory while the earlier approach aimed only to reach a final state without influencing the trajectory. The suitable class of reference trajectories is still restricted because they should go through a sequence of connected integral submanifolds. Highly oscillatory sequence (HOS) of inputs are built in the method and produce a sequence of trajectories converging to the reference trajectory. The major benefit using the new concept in comparison with the conventional approach emerges when obstacle avoidance is required in the configuration space. The method is demonstrated on simple model of the hexapod robot.

1 Introduction
One of the most important fields in control theory focuses on motion planning problem (MPP). The goal in this case is to determine an open loop control steering the system from an initial state to a desired final state. Several motion planning algorithms (MPAs) have been elaborated for kinematic systems, the most important ones in our framework are reported in [2], [3]. The first one reaches the final points along sequence of flows without influencing the shape of trajectory while the second one employs highly oscillatory sequence of inputs. There exist a typical class of systems where the smooth MPAs can not be directly applied. For instance, if intermittent constraints confine the system then different combinations of constraints define different submanifolds, each of them is equipped with different equations of motion. The phenomena leads typically nonsmooth equations of motion. The system equipped with this features is said to be stratified system. Such systems are mostly represented by walking robots. The basic theory of MP for stratified systems was established in [4]. It relies on the smooth MPA that uses sequence of flows because it makes the switching between two strata possible. The main drawback of this conventional technique appears in the unsolvable task of trajectory tracking.

A stratified approach that solves partly the trajectory tracking problem to nonholonomic stratified systems is proposed in this paper. The stratified system should satisfy a not too strict condition (DLARC). The notion of Decomposed Nonholonomic Representation (DNR) plays the central role in the new concept. The special structure of DNR is associated to a special class of reference trajectories called decomposed trajectories. Outline of finding a sequence of inputs for stratified systems that converges to a decomposed reference trajectory is given. The reference trajectory can be tracked by an approximation providing main benefit of the method in comparison to the previous ones because it can be used for obstacle avoidance.

The paper is organized as follows. The section 2 introduce the basic notions and reports the nonholonomic stratified MPA that is able to steer the stratified system along decomposed trajectory. Section 3 demonstrates nonholonomic stratified MPA on the simple model of hexapod robot and the appendix gives a brief overview about the stratified MP.

2 Decoupled Nonholonomic Representations (DNRs)
The conventional stratified MP discuss in [4] is briefly summarized in appendix. Depending on the
system nilpotency, it achieves or approximates a desired final configuration. The machinery reveals a main drawback, namely the trajectory tracking is not possible. This section is devoted to overcome this difficulty by building the concept onto a new representation of stratified systems called decoupled representation. The procedure below extensively use the results in [2], [4], and [3] to which the reader is referred for more details. From now on, the configuration space of bottom stratum will be denoted by $C_0$. This work regards in MPP for nonholonomic stratified systems which is (stratified) controllable.

A stratified controllable system is said to be nonholonomic stratified system (NSS) if it is not possible to find $n = \dim(C_0)$ independent moving on vector fields from the BSS. The main difficulty comes from stratification encumbering the use of all the vector fields at the same time.

**Definition 1.** Suppose, a distribution $\Delta$ is defined as $\Delta = \text{span}(g_1, \ldots, g_n)$. Then, one can define a function $\text{gen}(\Delta) = (g_1, \ldots, g_n)$ which is called the generator of the distribution $\Delta$.

Obviously, only vector fields define distributions in strict sense and a distribution does not provide vector fields uniquely. However, in this discussion, distributions are always related to explicit vector fields producing generator uniquely. Consider now our starting point, the bottom stratified system (BSS) [4]. It is useful to simplify its notations. For, let us denote a vector field $g_{i,j} |_{S_k}$ simply by $g_{i,j}$ omitting $|_{S_k}$ which is the $j$th vector field of the $i$th stratum considered in $S_k$ (where $|_{S_k}$ cuts off some dimensions if it is necessary). A stratum in previous works was defined by multiindex $I$. Since there are finite number of strata, multiindex $I$ may be replaced by simple integer $i$. Let $\Delta_i$ denote the distribution of the control vector fields in the $i$th stratum, i.e. $\Delta_i = \text{span}(g_{i,1}, \ldots, g_{i,d_i})$ where $d_i$ denotes the number of vector fields in the BSS from the $i$th stratum $S_i$.

**Definition 2.** Consider the involutive closures of $\Delta_i$ which is denoted by $\overline{\Delta}_i$. Then $\overline{\Delta}_i$ can be defined by $\overline{\Delta}_i = \text{span}(g_{i,1}, \ldots, g_{i,d_i})$, $\dim(\overline{\Delta}_i) = d_i$ where $g_{i,1}, \ldots, g_{i,d_i}$ are said to be the P. Hall generators of $\overline{\Delta}_i$ if they are the P. Hall basis of the Lie algebra $L(\text{gen}(\Delta_i)) \equiv L(g_{i,1}, \ldots, g_{i,d_i})$. P. Hall generators can be associated to a function $\text{pgen}$ where $(g_{i,1}, \ldots, g_{i,d_i}) = \text{pgen}(\Delta_i)$.

If $s$ denotes the number of strata which takes part in BSS then multiindex $I(l) = \underbrace{i_1 \ldots i_2}_{l \equiv |I| > s}$ is not allowed. The notation $I(l)(z) = t_z$ is used to point out a single element of the multiindex $I$.

**Definition 3.** Consider a BSS and let $s$ be the number of strata in the BSS. The sequence of involutive closures $\Delta_{I(l)} = \{\Delta_{i_1}, \ldots, \Delta_{i_l}\}, 1 \leq j \leq l, 1 \leq s \leq l$ is said to be a Decoupled Nonholonomic Representation (DNR) of the BSS.

**Definition 4.** Let be given a DNR $\overline{\Delta}_{I(l)} = \{\overline{\Delta}_{i_1}, \ldots, \overline{\Delta}_{i_l}\}, 1 \leq j \leq l, 1 \leq s$ where $\overline{\Delta}_{i_j} = \text{span}(g_{i_{j,1}}, \ldots, g_{i_{j,d_{i_j}}})$ i.e. $\text{gen}(\overline{\Delta}_{i_j}) = (g_{i_{j,1}}, \ldots, g_{i_{j,d_{i_j}}})$. Then the set of generators $\text{gen}(\overline{\Delta}_{I(l)}) = \{g_{i_{j,1}}, \ldots, g_{i_{j,d_{i_j}}}, \ldots, g_{i_{l,1}}, \ldots, g_{i_{l,d_{i_l}}}\}$ is said to be the fundamental generators of DNR.

**Definition 5.** Let be given a DNR $\Delta_{I(l)} = \{\Delta_{i_1}, \ldots, \Delta_{i_l}\}, 1 \leq j \leq l, 1 \leq s$ where $\Delta_{i_j} = \text{span}(g_{i_{j,1}}, \ldots, g_{i_{j,d_{i_j}}})$ i.e. $\text{gen}(\Delta_{i_j}) = (g_{i_{j,1}}, \ldots, g_{i_{j,d_{i_j}}})$. Then the set of generators $\text{gen}(\Delta_{I(l)}) = \{g_{i_{j,1}}, \ldots, g_{i_{j,d_{i_j}}}, \ldots, g_{i_{l,1}}, \ldots, g_{i_{l,d_{i_l}}}\}$ is said to be the extended generators of DNR.

It is seen that Lie brackets generated by vector fields of different strata (coupled Lie brackets) in DNR are not allowed.

**Definition 6.** Let us consider a DNR $\Delta_{I(l)} = \{\Delta_{i_1}, \ldots, \Delta_{i_l}\}, 1 \leq j \leq l, 1 \leq s$. The DNR satisfies the Decomposed Lie Algebra Rank Condition (DLARC) if $\sum_{j=1}^{l} \Delta_{i_j} = T_0 C_0$.

The condition of theorem 2 for NSS and the condition of DLARC in definition 6 (associated to DNR of BSS of NSS) are very similar. They are actually equivalent if the moving off vector fields are independent and decoupled from the moving on vector fields. This condition is supposed to be held during this section. The following statement gives a strict but sufficient condition for the DLARC.

**Proposition 1.** Consider the set of involutive closures $(\Delta_{i_1}, \ldots, \Delta_{i_l})$. If it is possible to find $n$ independent P. Hall generators from it then there exists DNR of BSS which satisfies DLARC.

**Definition 7.** Let a DNR of BSS be given in the form of $\Delta_{I(l)} = \{\Delta_{i_1}, \ldots, \Delta_{i_l}\}, 1 \leq j \leq l, 1 \leq s$. Moreover, let be given the initial and the final points $x_I = x(0)$ and $x_F = x(T)$. Then, a trajectory $x(t), 0 \leq t \leq T$ is said to be decomposed trajectory along $\Delta_{I(l)}$ between $x_I$ and $x_F$ if there exists a sequence of time intervals $t = \{t_1, \ldots, t_s\}$ such that $\sum_{j=1}^{s} t_j = T$ and $t < t_1 \Rightarrow \dot{x}(t) \in \Delta_{i_j}(x(t))$ and $\sum_{j=1}^{s} t_j \leq t \Rightarrow \dot{x}(t) \in \Delta_{i_j}(x(t)), 1 \leq j \leq l$.

**Definition 8.** A DNR of BSS in the form of $\Delta_{I(l)} = \{\Delta_{i_1}, \ldots, \Delta_{i_l}\}, 1 \leq j \leq l, 1 \leq s \leq l$ is said to be decomposed controllable if there exists decomposed trajectory along $\Delta_{I(l)}$ between any $x_I, x_F \in C_0$.

**Definition 9.** A DNR of BSS in the form of $\Delta_{I(l)} = \{\Delta_{i_1}, \ldots, \Delta_{i_l}\}, 1 \leq j \leq l, 1 \leq s \leq l$ is said to be weakly decomposed controllable if there exists
a sequence of input $u^j, j \to \infty$ that involves a sequence of trajectories $x^j(t)$ which converges to a decomposed trajectory $x^{dec}(t)$ along $\Delta_{I(1)}$ between any $x_I = x^{dec}(0), x_F = x^{dec}(T), x_I, x_F \in C_0$ such that $\|x_F - x^{\infty}(t)\| \leq \epsilon$ if $\|x_F - x_I\| < d_{\sigma}$ where $d_{\sigma}$ is a sufficiently small number for a given $\epsilon$. The $x^j(t)$ is denoted by $\bar{x}^{dec}(t)$ and said to be the approximation of the decomposed trajectory $x^{dec}(t)$.

Theorem 1. If $\Delta_{I(1)} = \{ \Delta_{i_1}, \ldots, \Delta_{i_s} \}, i_j \leq s, 1 \leq j \leq l, 1 \leq i \leq l$, satisfies DLARC then $\Delta_{I(1)}$ is weakly decomposed controllable.

Proof. The proof is constructive and provides an approximated decomposed trajectory along $\Delta_{I(1)}$ between any $x_I, x_F \in C_0, \|x_F - x_I\| < d_{\sigma}$. The proof proceeds as follows. Since DLARC holds, the set of involutive closures $\Delta_{i_1}, \ldots, \Delta_{i_s}$ spans the tangent space of $C_0$. As a consequence, it is possible to select $n$ independent vector fields $g_1, \ldots, g_n$ from the extended generators $\text{gen}(\Delta_{i_1}, \ldots, \Delta_{i_s})$. Let $g_1, \ldots, g_n$ be ordered according to the sequence of $\{ \Delta_{i_1}, \ldots, \Delta_{i_s} \}$. More precisely, for every indices $p$ and $q$, $1 \leq p < q, 1 \leq q \leq l$, if $p < q$ then $g_p \in \text{gen}(\Delta_{i_p}), g_q \in \text{gen}(\Delta_{i_q})$ where $\bar{p} \leq \bar{q}$. Based on this, if $k_j$ denotes the number of $\text{P. Hall}$ generators from $\Delta_{i_q}$ which takes part in $(g_1, \ldots, g_n)$, then $(g_{i_1}, \ldots, g_{i_1}, g_{i_1}, \ldots, g_{i_1}) =: (g_{i_1}, \ldots, g_{i_1})$. (It is easy to see that $\sum_{j=1}^l k_j = n$.)

Now, let it be supposed for a moment that $(g_{i_1}, \ldots, g_{i_1}, \ldots, g_{i_1})$ generate a nilpotent Lie Algebra. Then, the smooth MPA along flow sequence [2] is able to reach any $x_F = x(T) \in C_0$ from any $x_I = x(0) \in C_0$ as a sequence of flows. The resulted trajectory can be described in the form $x(T) = \Phi(g_{i_1}, t_1) \circ \cdots \circ \Phi(g_{i_1}, t_1) x(0)$ Let us define now $l + 1$ basis points in the following way. $b_0 = x(0) b_1 = \Phi(g_{i_1}, t_1) \circ \cdots \circ \Phi(g_{i_1}, t_1) b_2 = \Phi(g_{i_2}, t_2, k_2) \circ \cdots \circ \Phi(g_{i_{k_1}}, t_1, k_1) \circ \cdots \circ \Phi(g_{i_{k_1}}, t_1, k_1) x(0) 0 \leq t < \sum_{j=1}^{k_1} t_j$ satisfies $\hat{x}(t) \in \Delta_{i_1}$ since the flows evolve always along vector fields which belong to $\Delta_{i_1}$. As a result, the trajectory evolves on the integral submanifold of $\Delta_{i_1}$ passed through $x(0)$. Similarly, one can see that $\hat{x}(t) \in \Delta_{i_{k_1}}$, if $\sum_{j=1}^{k_1} t_j < t < \sum_{j=1}^{k_1+k_2} t_j$. This argument holds also for every subsection between two neighboring basis points concluding that $x(T) = \Phi(g_{i_1}, t_1, k_1) \circ \cdots \circ \Phi(g_{i_1}, t_1, k_1) \circ \cdots \circ \Phi(g_{i_1}, t_1, k_1) x(0)$ approximates a decomposed trajectory and leads the system from $x(0)$ to $x(T)$ where $\|x(T) - x_F\| < \epsilon$. As a consequence, the $\Delta_{I(1)}$ is weakly decomposed controllable and it proves the theorem.

It is worth remarking that basis points will lay in their strata after one use the real moving on vector fields. the reason is that one can always assume that the moving on vector fields are tangent to the bottom stratum [4] (if a moving on vector field is not tangent, one can make it tangent by using moving off vector fields). Note that every obstacle point is also projected into bottom stratum hence the results can be used for obstacle avoidance. Theorem 1 produces a trajectory along sequence of flows. Indeed, the trajectory was obtained by smooth MPA fitted to the basic stratified MP concept. However, keep in mind that smooth MP can not be applied to original stratified system but only to bottom BSS. The sequence of flows were selected in a special way which was defined by the DNR created from BSS. The advantage of such a structured sequence turns out bellow (cf. corollary 1).

Corollary 1. Let be considered a DNR $\Delta_{I(1)} = \{ \Delta_{i_1}, \ldots, \Delta_{i_s} \}, i_j \leq s, 1 \leq j \leq l, l \leq s$ satisfying DLARC. Let a MPP be given between $x_I$ and $x_F$. Moreover, let the reference trajectory be an $\bar{x}_d(t)$ approximation of decomposed trajectory $x_d(t)$ along $\Delta_{I(1)}$ equipped with the basis points in theorem 1. Then, there exists a sequence of inputs $u^j(t), j \to \infty$ that generates a sequence of trajectories $x^j(t)$ which converges to $\bar{x}_d(t)$.

The proof of this is can be found in [1], [5]. Based on the results, the nonholonomic stratified MP along decomposed trajectories is summarized as follows.

Algorithm 1. (Nonholonomic stratified MP with decomposed trajectory approximation)

Step 1. Create the BSS of stratified system (5)
Step 2. Obtain $\Delta_1, \ldots, \Delta_s$.
Step 3. Determine the involutive closures $\Delta_j$ and their P. Hall generators.
Step 4. Create a DNR $\Delta_{I(1)}$ that satisfies DLARC. Use proposition 1.
Step 5. Obtain the fundamental and extended generators of $\Delta_{I(1)}$ (see definition 4 and 5).
Step 6. Obtain the basis points $b_0, \ldots, b_t$ by using theorems 1.
Step 7. Plan the reference decomposed trajectory along $\Delta_{I(1)}$ which goes through $b_0, \ldots, b_t$.
Step 8. Produce a highly oscillatory sequence (HOS) of inputs $u^j$ that generates a sequence of trajectories $x^j(t)$ Use the procedure of theorem 1 so that $x^2(t)$ converges to the prescribed decomposed trajectory.
3 Illustrative Example

This section demonstrates the method on the example of hexapod robot as seen in Fig. 1. In the sequel, \(x, y\) denote the position of the robot in the plane, \(\theta\) is the orientation, \(\phi_1\) and \(\phi_2\) are the angles of the leg groups. The height of the leg groups are denoted by \(h_1\) and \(h_2\). The \(l\) stands for the length of the legs (for the sake of easier computations we assume \(l = 1\)). The system has \(s = 3\) strata whose equations of motion are given as follows. On the stratum \(S_0\), all legs are on the ground, the state vector is \(x_0 = (\dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}_1, \dot{\phi}_2)^T\) and the equation of motion is given by

\[
\dot{x}_0 = \begin{pmatrix} \cos(\theta), \sin(\theta), 1, 1, 0 \end{pmatrix}^T u_1 + \begin{pmatrix} \cos(\theta), \sin(\theta), -1, 0, 1 \end{pmatrix}^T u_2 = [g_{01}, g_{02}] [u_1 \ u_2]^T. \tag{1}
\]

On the stratum \(S_1\), the leg group 1 is in the air, the state vector is \(x_1 = (\dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}_1, \dot{\phi}_2, h_1)^T\) and the equation of motion is given by

\[
\dot{x}_1 = \begin{pmatrix} \cos(\theta), \sin(\theta), 1, 1, 0, 0 \end{pmatrix}^T u_1 + \begin{pmatrix} 0, 0, 0, 0, 1, 0 \end{pmatrix}^T u_2 + \begin{pmatrix} 0, 0, 0, 0, 1, 0 \end{pmatrix}^T v_4 = [g_{11,1} g_{12,1}] [u_1 \ u_2 \ u_4]^T. \tag{2}
\]

On the stratum \(S_2\), the leg group 2 is in the air, the state vector is \(x_2 = (\dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}_1, \dot{\phi}_2, h_1)^T\) and the equation of motion is given by

\[
\dot{x}_2 = \begin{pmatrix} \cos(\theta), \sin(\theta), -1, 0, 1, 0 \end{pmatrix}^T u_1 + \begin{pmatrix} 0, 0, 0, 1, 0, 1 \end{pmatrix}^T u_2 + \begin{pmatrix} 0, 0, 1, 0, 0 \end{pmatrix}^T u_4 + \begin{pmatrix} 0, 0, 0, 0, 1, 0 \end{pmatrix}^T v_4 = [g_{21,2} g_{22,2}] [u_1 \ u_2 \ u_4]^T. \tag{3}
\]

Let the MPP consist of finding input that steers the hexapod robot from \(x_I = (0, 0, 0, 0, 0)^T\) to \(x_F = (0, 0, 0, 0, 0)^T\) in the bottom stratum. The MP is performed by the steps of algorithm 1.

Step 1. Create the BSS. The differential equation of BSS is depicted in the form \(\Sigma^{BSS} : \dot{x} = g_{01}(\phi_1^0, 1) + g_{02}(\phi_2^0, 2) \begin{pmatrix} s_1 \ u_1, 1^2 + g_{21,2} \begin{pmatrix} \cos(\theta), \sin(\theta), 1, 1, 0 \end{pmatrix}^T u_1, \begin{pmatrix} \cos(\theta), \sin(\theta), -1, 0, 1 \end{pmatrix}^T u_2 \end{pmatrix} = [g_{01,1}, g_{02,1}] [u_1 \ u_2]^T. \tag{3}
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which generate a Lie algebra with maximal rank. Let them be \( g_{1,1}, g_{1,2}, g_{2,2}, g_{3,2} \). The solution is obtained in the form of sequence of flows

\[ x(T) = \Phi(-1.24g_{1,2}, 0.16) \circ \Phi(-1.24g_{2,2}, 0.16) \circ \]
\[ \Phi(-1.24g_{1,1}, 0.33) \circ \Phi(-1.24g_{1,1}, 0.33) \circ \]
\[ \Phi(1.24g_{1,2}, 0.33) \circ \Phi(1.24g_{1,1}, 0.33) \circ \Phi(1.24g_{1,1}, 0.16) \circ \]
\[ \Phi(1.24g_{1,1}, 0.16)x_F. \]  

The generated Lie algebra is not nilpotent, hence the \( x(T) \) does not exactly coincide with \( x_F \). Namely, \( x(T) = (0.39 \ 0.39 \ 0 \ 0 \ 0)^T \). From this the basis points of the smooth MP are

\[ b_0 = x_I = (0 \ 0 \ 0 \ 0 \ 0)^T \]
\[ b_1 = \Phi(-1.24g_{1,2}, 0.33) \circ \Phi(-1.24g_{1,1}, 0.33) \]
\[ \circ \Phi(1.24g_{1,2}, 0.33) \circ \Phi(1.24g_{1,1}, 0.33) \]
\[ \circ \Phi(1.24g_{1,1}, 0.16) \circ \Phi(1.24g_{1,1}, 0.16)b_0 \]
\[ = (0.39 \ 0.39 \ 0 \ 0.2 \ 0.2)^T \]
\[ b_2 = \Phi(-1.24g_{2,2}, 0.16)b_1 = (0.39 \ 0.39 \ 0 \ 0.2 \ 0)^T \]
\[ b_3 = \Phi(-1.24g_{3,2}, 0.16)b_2 = (0.39 \ 0.39 \ 0 \ 0 \ 0)^T \]

Step 7. Plan a decomposed trajectory along \( \Delta_{1(1,2,3)} \) which goes through \( b_0, \ldots, b_3 \). Let the trajectory between \( b_1 \) and \( b_2 \) be the same as determined in previous step and let us concentrate now only to the trajectory between \( b_0 \) and \( b_1 \) which gives the path of the hexapod robot in the plane. Let a straightline trajectory be desired between \( b_0 \) and \( b_1 \). The Figure 2 shows the resulted trajectory with the earlier stratified motion planning where the trajectory between \( b_0 \) and \( b_1 \) can not be prescribed.

Step 8. Producing a highly oscillatory sequence (HOS) of inputs \( u^j \) that generates a sequence of trajectories \( x^j(t) \). Use the procedure in [3] so that \( x^j(t) \) converge to the reference straightline trajectory between \( b_0 \) and \( b_1 \). One can see, that a straightline trajectory can be produced by \( g_{1,1}, g_{1,2} \) and \( g_{1,3} \) without using \( g_{1,4} \). In fact, [3] showed that \( u^j(t) = \sqrt{j} \cos(jt)g_{1,1} + \sqrt{j} \sin(jt)g_{1,2} \) converge to \( g_{1,3}/2 \) which can be easily performed. The straightline trajectory depicted in Figure 2. The feature of the input is illustrated in a smaller time interval on Figure 3. In the algorithm, \( j = 500 \) was applied. Of course, \( j \) is far from infinity but still provided good approximation.

Step 9. Insert moving off vector fields at the basis points following the stratified MPA [4].

Figure 3: The highly oscillatory inputs that generates a straightline path for hexapod robot in the plane.

4 Conclusion

The foundation of a nonholonomic stratified MPA using decomposed trajectories was introduced. It is supposed that stratified system possesses DLARC property. The proposed technique enhances the efficiency of basic stratified MP in trajectory tracking since it is able to follow a special class of trajectories while the earlier approach could reach only a desired final point without influencing the shape of trajectory. The extra feature may be exploited when obstacle avoidance is necessary in the bottom stratum. (It is possible to project other obstacle points from higher strata into this stratum). A trajectory tracking problem on the simple example of hexapod robot was also considered and solved in the paper. Difficulties may occur in some applications depending on the symbolic complexity of the vector fields and their operations and the real realization problem of the highly oscillatory inputs.

5 Appendix: Summary of stratified MP

In this section, we summarize the theoretical background of stratified MP [4]. This theory intensively relies on the smooth MP due to Lafferriere and Sussmann [2]).
Definition 10. A set \( K \subset \mathbb{R}^n \) defined by union of smooth manifolds (i.e. strata) is said to be regularly stratified set.

Definition 11. The system is stratified if its configuration space is defined by regularly stratified sets.

Let \( S_0 \equiv M \) be the entire configuration space where there is no constraint. Let the stratum \( S_i \subset S_0 \), \( i > 0 \) be a codimension one submanifold where the system is subjected to a kinematic constraint. Roughly speaking, this stratum corresponds to dimension \( n - 1 \) manifold in the configuration space. Let \( S_{i,j} = S_i \cap S_j \) where system is subjected to the two constraints presented on \( S_i \) and \( S_j \). In general, a stratum where a few constraints may appear is denoted by \( S_I = S_{i_1}\ldots i_k = S_{i_1} \cap \cdots \cap S_{i_k} \) where \( I = i_1i_2\ldots i_k \) is a multi-index. The stratum with lowest dimension is said to be the bottom stratum. A stratum is called lower stratum if its dimension is lower than the dimension of the other one. The higher stratum is defined vice versa.

Theorem 2. (Goodwine) Let \( T_{x_0}M \) be the tangent space of \( M \) in \( x_0 \) and let \( \Delta_{S_j} |_{x_0} \) denote the involutive closures of a distribution which is spanned by the vector fields of a stratum \( S_j \) in \( x_0 \). If there exists a nested sequence of strata \( x_0 \in S_0 \subset S_{p-1} \subset \cdots \subset S_1 \subset S_0 \), such that the involutive closures of distributions (of strata) fulfill \( \sum_{j=0}^{p} \Delta_{S_j} |_{x_0} = T_{x_0}M \) then the system is locally stratified controllable from \( x_0 \).

Definition 12. A vector field is said to be a moving off vector field if the existence of contact points depends on it. In other words, the moving off vector fields switches between strata.

Definition 13. A vector field is said to be a moving on vector field if it does not leave the actual stratum.

The stratified MPA can be summarized as follows.

Algorithm 2. [Stratified Motion Planning]

Step 1. Determine the multiple stratified system.

The equation of motion on the stratum \( S_0 \) is \( \dot{x} = g_0.1u^{0,1} + \cdots + g_0.n_0u^{0,n_0} \). Similarly, the equations of motion are \( \dot{x} = g_0.1u^{0,1} + \cdots + g_0.n_0u^{0,n_0} \) on \( S_1 \), and \( \dot{x} = g_{I,n_1}u^{I,n_1} + \cdots + g_{I,n_1}u^{I,n_1} \) on \( S_I \).

Step 2. Create a bottom stratified system (BSS). It is clear from Theorem 2 that the stratified MPA demands a special system on the bottom stratum (as a common space). Note, this system is also a fictitious system but without any connection to the other fictitious system discussed by the smooth MP. In order to obtain this special (i.e. bottom stratified system), one has to create a union set of vector fields in all strata and define with them a system \( \Sigma^{BSS} \)

\[
\dot{x} = g_{0.1}u^{0,1} + \cdots + g_{0,n_0}u^{0,n_0} + g_{I,n_1} |_{S_0} u^{1,1} + \cdots \\
+ g_{I,n_1} |_{S_0} u^{1,n_1} + g_{I,1} |_{S_0} u^{1,1} + \cdots \\
+ g_{I,n_1} |_{S_0} u^{1,n_1} 
\]

(5)

where the notation \( |_{S_0} \) refers to the vector fields which take a part in bottom stratified system, however, they are defined originally not in this stratum. It should consist of all the moving on vector fields that commute with the moving off vector fields (i.e. the vector fields which disconnect the contact between the legs and the ground). Note, that \( S_0 \) does not denote the whole configuration space in this description but the bottom stratum.

Step 3. Create the bottom stratified extended system \( \Sigma^{BSS} = \Sigma + \text{Lie brackets including the vector fields of bottom stratified system and the Lie brackets among them}. \)

Step 4. Solve the smooth MP [2] on the bottom stratified extended system. The result is a sequence of flows along the moving on vector fields.

Step 5. Complete the solution for the stratified MPP. If two neighboring flows in the sequence are defined in different strata, one should insert moving off vector fields between them which switch also between their strata. The extra flows along the moving off vector fields have to assure that the set of constraints changes in accordance with the change of strata. It is satisfied from the conception.

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References


